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Convergence of the Ishikawa Iteration Process for Nonexpansive Mappings

Lei Deng

Department of Mathematics, Southwest China Normal University, Beibei, Chongqing 630715, People's Republic of China

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Let D be a subset of a normed space X and $T\colon D \to X$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in D and two real sequences $\{t_n\}$ and $\{s_n\}$ satisfying

(i) $0 \le t_n \le t < 1$ and $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) $0 \le s_n \le 1$ and $\sum_{n=1}^{\infty} s_n < \infty$,
(iii) $x_{n+1} = t_n T(s_n T x_n + (1 - s_n) x_n) + (1 - t_n) x_n$, $n = 1, 2, 3, \ldots,$

we prove that if $\{x_n\}$ is bounded, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The conditions on D, X, and T are shown which guarantee the weak and strong convergence of the Ishikawa iteration process to a fixed point of T.

1. INTRODUCTION

Let D be a subset of a normed space X and $T\colon D \to X$ be nonexpansive (this is, $\|Tx - Ty\| \le \|x - y\|$ for all x, y in D). In 1974, Ishikawa [5] introduced a new iteration process as

$$x_{n+1} = t_n T\big(s_n T x_n + (1 - s_n) x_n\big) + (1 - t_n) x_n, \qquad n = 1, 2, 3, \ldots,$$

where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0, 1]$ satisfying certain restrictions. The Mann iteration process is a special case of Ishikawa where $s_n = 0$ for all $n \ge 1$ [1, 4, 7].

In 1976, Ishikawa [6] proved the following theorem without any assumption on convexity of the Banach space.

THEOREM A [6]. *Let D be a subset of a normed space X and $T\colon D \to X$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in D and a sequence $\{t_n\}$*

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of real numbers satisfying

(i) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $x_{n+1} = (1 - t_n)x_n + t_n T x_n$, *for* $n = 1, 2, 3, \dots$,

if $\{x_n\}$ *is bounded, then* $\|Tx_n - x_n\| \to 0$ *as* $n \to \infty$.

In this paper, we first extend Theorem A to the Ishikawa iteration process. Then we show the weak and strong convergence of the Ishikawa iteration process in a Banach space. Our results generalize the results of Emmauele [2] and Ishikawa [6].

2. LEMMAS

LEMMA 1 [11]. *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are two sequences of nonnegative numbers such that* $a_{n+1} \le a_n + b_n$ *for all* $n \ge 1$. *If* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists*.

LEMMA 2. *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are two sequences of a normed space* X. *If there is a sequence* $\{t_n\}$ *of real numbers satisfying*

(i) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $a_{n+1} = (1 - t_n)a_n + t_n b_n$ *for all* $n \ge 1$

(iii) $\lim_{n\to\infty} \|a_n\| = d$,

(iv) $\limsup_{n\to\infty} \|b_n\| \le d$ *and* $\{\sum_{i=1}^{n} t_i b_i\}$ *is bounded,*

then $d = 0$.

Proof. Suppose that $d > 0$ and it follows from (iv) that $\sum_{i=n}^{n+m-1} t_i b_i$ is bounded for all n and m. Let

$$M = \sup\left\{\left\|\sum_{i=n}^{n+m-1} t_i b_i\right\| : n, m = 1, 2, 3, \dots\right\}.$$

Choose a number N such that

$$N > \max\left\{\frac{2M}{d}, 1\right\}.$$

We can choose a positive ε such that

$$1 - 2\varepsilon \exp\left(\frac{N+1}{1-t}\right) > \frac{1}{2}.$$

It follows from (i) that there exists a natural k such that

$$N < \sum_{i=1}^{k} t_i \le N + 1.$$

Since $\lim_{n\to\infty} \|a_n\| = d$, $\limsup_{n\to\infty} \|b_n\| \le d$ and ε independent of n, without loss of generality we may assume that for all $n \ge 1$,

$$d(1-\varepsilon) < \|a_n\| < d(1+\varepsilon) \qquad \text{and} \qquad \|b_n\| < d(1+\varepsilon).$$

Setting $T = \sum_{i=1}^{k} t_i$, $S = \prod_{i=1}^{k} s_i$ and $s_n = 1 - t_n$ for all $n \ge 1$, from (ii), we obtain

$$a_{k+1} = s_1 s_2 \dots s_k a_1 + t_1 s_2 s_3 \dots s_k b_1 + \cdots + t_{k-1} s_k b_{k-1} + t_k b_k,$$

$$a_{k+1} \in B := \operatorname{co}\{a_1, b_1, b_2, \dots, b_k\}.$$

Let $x = T^{-1} \sum_{i=1}^{k} t_i b_i$ and

$$y = S(1-S)^{-1}\big\{a_1 + t_1\big(s_1^{-1} - T^{-1}\big)b_1 + t_2\big(s_1^{-1}s_2^{-1} - T^{-1}\big)b_2 + \cdots + t_k(S^{-1} - T^{-1})b_k\big\}.$$

Then it is clear that $x, y \in B$ and $a_{k+1} = Sx + (1-S)y$. Therefore

$$\begin{aligned} d(1-\varepsilon) < \|a_{k+1}\| &\le S\|x\| + (1-S)\|y\| \\ &\le S\|x\| + (1-S)d(1+\varepsilon). \end{aligned}$$

Hence, we have

$$\begin{aligned} \|x\| > d\big(1 - S^{-1}(2-S)\varepsilon\big) &> d(1 - 2\varepsilon S^{-1}) \\ &= d\left(1 - 2\varepsilon \prod_{i=1}^{k} (1-t_i)^{-1}\right) \\ &= d\left[1 - 2\varepsilon \exp\left(\sum_{i=1}^{k} \log\left(1 + \frac{t_i}{1-t_i}\right)\right)\right] \\ &\ge d\left(1 - 2\varepsilon \exp\left(\sum_{i=1}^{k} \frac{t_i}{1-t_i}\right)\right) \\ &\ge d\left(1 - 2\varepsilon \exp\left(\frac{T}{1-t}\right)\right) \\ &\ge d\left(1 - 2\varepsilon \exp\left(\frac{N+1}{1-t}\right)\right) > \frac{d}{2}, \end{aligned}$$

since $\log(1+u) \le u$ for $-1 < u < \infty$.

On the other hand, we have

$$\|x\| = T^{-1}\left\|\sum_{i=1}^{k} t_i b_i\right\| \le T^{-1}M \le \frac{d}{2M}M = \frac{d}{2},$$

arriving at a contradiction. This completes the proof.

LEMMA 3 [11]. *Let D be a subset of a normed space X and $T\colon D \to X$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in D satisfying*

$$x_{n+1} = t_n T\big(s_n T x_n + (1-s_n)x_n\big) + (1-t_n)x_n, \qquad n = 1,2,3,\ldots,$$

where $0 \le t_n,\ s_n \le 1$ for all $n \ge 1$, then

$$\|x_{n+1} - p\| \le \|x_n - p\| \qquad \text{for all } n \ge 1 \text{ and all } p \in F(T),$$

where $F(T)$ denotes the set of fixed points of T.

3. MAIN RESULTS

THEOREM 1. *Let D be a subset of a normed space X and $T\colon D \to X$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in D and two real sequences $\{t_n\}$ and $\{s_n\}$ satisfying*

(i) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $0 \le s_n \le 1$ *and* $\sum_{n=1}^{\infty} s_n < \infty$,

(iii) $x_{n+1} = t_n T(s_n T x_n + (1-s_n)x_n) + (1-t_n)x_n$, $n = 1,2,3,\ldots,$

if $\{x_n\}$ is bounded, then $\|Tx_n - x_n\| \to 0$ as $n \to \infty$.

Proof. Set $y_n = s_n T x_n + (1-s_n)x_n$, then $x_{n+1} = t_n T y_n + (1-t_n)x_n$. By page 304 of [11], we have

$$\|Tx_{n+1} - x_{n+1}\| \le \big(1 + 2s_n(1-t_n)\big)\|Tx_n - x_n\|.$$

Setting $a_n = Tx_n - x_n$, by (ii) and the fact that $\{\|Tx_n - x_n\|\}$ is bounded, it follows from Lemma 1 that $\lim_{n\to\infty}\|a_n\| = d$.

We may assume that $t_n > 0$ for all $n \ge 1$, without loss of generality. Setting $b_n = t_n^{-1}(Tx_{n+1} - Tx_n) + Tx_n - Ty_n$, we have $a_{n+1} = (1-t_n)a_n + t_n b_n$ and

$$\begin{aligned}
\|b_n\| &\le t_n^{-1}\|Tx_{n+1} - Tx_n\| + \|Tx_n - Ty_n\| \\
&\le t_n^{-1}\|x_{n+1} - x_n\| + \|x_n - y_n\| \\
&= \|Ty_n - x_n\| + s_n\|Tx_n - x_n\| \\
&\le \|Ty_n - Tx_n\| + \|Tx_n - x_n\| + s_n\|Tx_n - x_n\| \\
&\le \|Tx_n - x_n\| + 2s_n\|Tx_n - x_n\|.
\end{aligned}$$

By (ii), $\lim_{n\to\infty} s_n = 0$, we have

$$\limsup_{n\to\infty} \|b_n\| \le d.$$

Finally, we have

$$\begin{aligned}\left\| \sum_{i=1}^{n} t_i b_i \right\| &= \left\| \sum_{i=1}^{n} \left(Tx_{i+1} - Tx_i + t_i(Tx_i - Ty_i)\right) \right\| \\ &= \left\| Tx_{n+1} - Tx_1 + \sum_{i=1}^{n} t_i(Tx_i - Ty_i) \right\| \\ &\le \|Tx_{n+1} - Tx_1\| + \sum_{i=1}^{n} t_i \|Tx_i - Ty_i\| \\ &\le \|x_{n+1} - x_1\| + \sum_{i=1}^{n} t_i s_i \|Tx_i - x_i\|\end{aligned}$$

which is bounded. Hence it follows from Lemma 2 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, completing the proof.

Theorem 1 above generalizes Lemma 2 of Ishikawa [6].

Recall that a Banach space X is said to satisfy Opial's condition [8] if the condition $x_n \to x_0$ weakly implies

$$\limsup_{n\to\infty} \|x_n - x_0\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \neq x_0$.

THEOREM 2. *Let X be a Banach space which satisfies Opial's condition, D be weakly compact, and let T and $\{x_n\}$ be as in Theorem* 1. *Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Due to weak compactness of D, there exists $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a $p \in D$. With standard proof we show that $p = Tp$. We suppose that $\{x_n\}$ doesn't converge weakly to p; then there are $\{x_{n_j}\}$ and $q \neq p$ such that $x_{n_j} \to q$ weakly and $q = Tq$. By Lemma 3 and Opial's condition of X, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\ &= \lim_{j\to\infty} \|x_{n_j} - q\| < \lim_{j\to\infty} \|x_{n_j} - p\| \\ &= \lim_{n\to\infty} \|x_n - p\|,\end{aligned}$$

a contradiction. This completes the proof.

Theorem 2 above generalizes a result of Emmanuele [2].

Let D be a subset of a Banach space X. A mapping $T\colon D \to X$ with a nonempty fixed points set $F(T)$ in D will be said to satisfy Condition A [10] if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$, such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in D$, where $d(x, F(T)) = \inf\{\|x - z\|\colon z \in F(T)\}$.

The two following theorems generalize Theorems 1 and 2 of Ishikawa [6], respectively. Since a similar proof is in [6, 11], we omit their proof here.

THEOREM 3. *Let D be a closed subset of a Banach space X, and T be a nonexpansive mapping from D into a compact subset of X. If $\{x_n\}$ is as in Theorem* 1, *then $\{x_n\}$ converges to a fixed point of T.*

THEOREM 4. *Let X, D, and $\{x_n\}$ be as in Theorem* 3. *Let $T\colon D \to X$ be a nonexpansive mapping with a nonempty fixed points set $F(T)$ in D. If T satisfies Condition* A, *then $\{x_n\}$ converges to a member of $F(T)$.*

The following theorem generalizes Theorem 1.8 of Veeramani [12].

THEOREM 5. *Let D be a closed, bounded, convex subset of a Banach space X and $T\colon D \to D$ be a map satisfying*

(i) $\|Tx - Ty\| \leq \|x - y\|$, *for all* $x, y \in D$;

(ii) *For some* $\alpha > 0$,

$$\|Tx - Ty\| \leq \alpha(\|x - Tx\| + \|y - Ty\|), \qquad \text{for all } x, y \in D.$$

If $\{x_n\}$ is as in Theorem 1, *then $\{x_n\}$ converges to the unique fixed point of T.*

Proof. By Gillespie and Williams [3], it follows that T has a unique fixed point p in D. From (ii), we have

$$\|Tx_n - p\| = \|Tx_n - Tp\| \leq \alpha\|x_n - Tx_n\|.$$

Hence, by Theorem 1, we obtain

$$\|Tx_n - p\| \to 0 \qquad \text{as } n \to \infty.$$

It follows that

$$\|x_n - p\| \leq \|x_n - Tx_n\| + \|Tx_n - p\| \to 0 \qquad \text{as } n \to \infty.$$

This completes the proof.

The following theorem extends Theorem 2 of Reich [9] in uniformly convex Banach spaces.

THEOREM 6. *Suppose that X is a uniformly convex Banach space with a Frechet differentiable norm, C a nonempty closed convex subset of X,*

$T: C \to C$ a nonexpansive mapping with a fixed point, and $\{x_n\}$ is as in Theorem 1. Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Let $\omega_w(x_n)$ be the weak limit ω-set of $\{x_n\}$, i.e., the set $\{x \in X: x = \text{weak-lim}_{k\to\infty} x_{n_k}$ for some $n_k \uparrow \infty\}$. By Theorem 1 and Browder's demiclosedness principle, $\omega_w(x_n)$ is contained in $F(T)$, the fixed point set of T.

The remainder of the proof is similar to that of Theorem 1 of [11], so the details are omitted.

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